INFINITESIMAL TORELLI THEOREM FOR CYCLIC COVERINGS OF GENERALIZED FLAG VARIETIES I

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Abstract. We give an effective infinitesimal Torelli theorem for cyclic covers of $\mathbf{G}/\mathbf{P}$ where $\mathbf{G}$ is a simple algebraic group and $\mathbf{P}$ is a maximal parabolic subgroup.

Introduction

The Torelli problem for a given family of varieties ask wheter varieties of the family can be distinguished by their Hodge structures. It is known to be the case for curves [17], [1] and [19], some K3 surfaces [14] and Prym varieties [13].

There are some variants that have been studied: whether the period map is an immersion (local Torelli), wheter its differential is injective in the deformation space (infinitesimal Torelli) and whether the map is generically injective (generic Torelli). Generic Torelli is known to hold for hypersurfaces of degree d in the projective space (except for few exceptions, see [8]), while the infinitesimal Torelli is known for hypersurfaces of high degree of $\mathbf{G}/\mathbf{P}$ [7].
Let X be a generalized flag variety $\mathbf{G}/\mathbf{P}$, with $\mathbf{G}$ a simple algebraic group and $\mathbf{P}$ a maximal parabolic subgroup, embedded minimally and equivariantly into a projective space by an ample line bundle $\mathcal{O}_X(1)$ (which generates the Picard group of X in this case) and let $\Omega^q(k) = \Omega^q \otimes \mathcal{O}_X(1)^k$ be the sheaf of holomorphic $q-$forms on X tensored with the k^{th} power of $\mathcal{O}_X(1)$.

Let Z be a simple covering of X of degree N, in particular Z can be embeded in $\mathrm{Spec}_{\mathcal{O}_X}(\oplus_{i=0}^{\infty}\mathcal{L}^{-i})$ for some ample line bundle $\mathcal{L} = \mathcal{O}(k)$. Following some ideas of Flenner [9], Green [8] and Ivinskis [12], we want to show that the infinitesimal Torelli theorem holds for Z under certain effective conditions (see 5.2) on the degree of the covering and the degree of $\mathcal{L}$.

The infinitesimal Torelli theorem ask about the injectivity of the tangent map

$$T : H^1(Z, \tau_Z) \to T_{m(0)}D = \oplus_p \mathrm{Hom}(H^{n-p}(Z, \Omega_Z^p), H^{n-p+1}(Z, \Omega_Z^{p-1}))$$

(see below), but this can be reduced (as in [12]) to the vanishing of some cohomology groups on the base space X, which can be done in our case using an idea of Deligne-Dimca [5], the results of Snow [15]and [16] as well as a classical theorem of Bott [3].

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More precisely, let B be the Kuranishi space of deformations of Z. Suppose B is smooth and let $m : B \to D$ be the local period map into the period domain D of all Hodge structures on $H^k(Z, \mathbb{C})$, where $k = \dim Z$. The map m is holomorphic, with tangent map at the point Z:

$$T : H^1(Z, \tau_Z) \to T_{m(0)}D$$

We say that the infinitesimal Torelli theorem holds for X if the tangent map T is injective.

The paper is organized as follows: In section 1 we remember some basic facts about generalized flag varieties with Picard group $\mathbb{Z}$, in particular its classification. In section 2 we recall some vanishing theorems for the varieties corresponding to Lie Groups of types A, B, C, D, E_6 and E_7. Section 3 reviews the definition of a simple cover and some basic facts about line bundles on them. In section 4 we prove the Infinitesimal Torelli theorem for cyclic covers of $\mathbf{G}/\mathbf{P}$ under the assumptions above (i.e., $Pic(\mathbf{G}/\mathbf{P}) \cong \mathbb{Z}$).

1. Generalized flag varieties having Picard group $\mathbb{Z}$

Any generalized flag variety is of the form $X := \mathbf{G}/\mathbf{P}$ with $\mathbf{G}$ a simple algebraic group and $\mathbf{P}$ a parabolic proper subgroup. If $\mathbf{T}$ is a maximal torus of $\mathbf{G}$, one associates to it a set Φ of characters of the torus, called the *root system* of $\mathbf{G}$ relative to $\mathbf{T}$ (or just the root system of $\mathbf{G}$), together with a base $\Delta \subset \Phi$ of the vector space generated by the characters of the torus.

Since $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ is a base of the vector space generated by the characters of $\mathbf{T}$, then any root $\alpha \in \Phi$ can be writen as $\alpha = \sum_i a_i \alpha_i$. In particular a root is said to be positive if $a_i \geq 0$ for all i. The set of positive roots is denoted as Φ^+ and one has $\Phi = \Phi^+ \cup \Phi^-$, where $\Phi^+ = -\Phi^-$.

The tangent space to $\mathbf{G}$ at the identity is a Lie algebra g and the tangent space to $\mathbf{P}$ at the identity is a Lie subalgebra p. If $\mathbf{T}$ is a maximal torus on $\mathbf{G}$ contained in $\mathbf{P}$, then

$$g = t \oplus \left(\bigoplus_{\alpha \in \Phi^-} g_\alpha \right) \oplus \left(\bigoplus_{\alpha \in \Phi^+} g_\alpha \right)$$

and

$$p = t \oplus \left(\bigoplus_{\alpha \in \Phi^-} g_\alpha \right) \oplus \left(\bigoplus_{\alpha \in \Phi_c} g_\alpha \right)$$

where t is the Lie algebra associated to the torus T, g_α is a one dimensional vector space associated to the root α and Φ_c is a subset of Φ^+ which depends on $\mathbf{P}$, called the set of *positive compact roots*. In particular it implies that the tangent space of $\mathbf{G}/\mathbf{P}$ at any point is isomorphic to

$$\tau = \bigoplus_{\alpha\in\Phi^+-\Phi_c} g_\alpha$$

The parabolic group $\mathbf{P}$ is said to be generated by $A \subset \Delta = \{\alpha_1, \cdots, \alpha_l\}$ if

$$\Phi_c = \{\alpha = \sum_i a_i\alpha_i \in \Phi^+ \mid a_j = 0 \text{ for all } \alpha_j \in A\}$$

Any parabolic group is generated by some $A \subset \Delta$. If A consist of a single element, then we say that P is a *maximal parabolic subgroup* generated by that element. Observe that

$$\tau = \bigoplus_{\alpha\in\Phi^+-\Phi_c} g_\alpha$$

where $\Phi^+ - \Phi_c$ consists of the positive roots for which the coefficient of α_j is strictly positive, for every $\alpha_j \in A$.

The simple algebraic groups are completely classified and they have type: A_l, B_l, C_l, D_l, E_6, E_7, E_8, F_4 and G_2. We will only consider maximal parabolic subgroups and the clasical groups (A_l, B_l, C_l and D_l) in this paper, but the results are also true for the special groups as well.

1.1) If the type of $\mathbf{G}$ is A_l, then $\mathbf{G} = SL(l+1)$, $\mathbf{P}$ is generated by $\{\alpha_r\}$ for some $1 \leq r \leq l+1 = r+t$ and $X = \mathbb{G}r(r,t)$.

1.2) If the type of If the type of $\mathbf{G}$ is B_l, $l \geq 2$, then $\mathbf{G} = SO(2l+1)$.

1.3) If the type of $\mathbf{G}$ is C_l, then $\mathbf{G} = Sp(2l)$.

1.4) If the type of $\mathbf{G}$ is D_l, then $\mathbf{G} = SO(2l)$.

The following is a well known fact on the theory of algebraic groups, see for instance ([2], section 11) or ([11], section 16).

Lemma 1.1. *Let $\mathbf{G}, \mathbf{P}$ and $\mathbf{T}$ be as before, and let $N_{\mathbf{G}}(\mathbf{T})$ and $C_{\mathbf{G}}(\mathbf{T})$ be the normalizer and the centralizer of $\mathbf{T}$ on $\mathbf{G}$ respectively. Then the group*

$$W := \frac{N_{\mathbf{G}}(\mathbf{T})}{C_{\mathbf{G}}(\mathbf{T})}$$

is finite. It is called the Weyl group of $\mathbf{G}$ (relative to $\mathbf{T}$).

Remark 1.2. *There is a well defined notion of length for the elements of the Weyl Group, which we do not want to make precise here because it will not be really necessary for what comes next. Let us just mention that, in the case of groups of type A_l, the Weyl group is the group of permutations S_l and, in this case, the length of an element is the usual length of a permutation.*

2. The cohomology of $\Omega^q_X(k)$

Associated to every root $\alpha_i \in \Delta$ there exist a fundamental weight λ_i wich satisfies $\dfrac{2(\alpha_i, \lambda_j)}{(\alpha_i, \alpha_j)} = \delta_{i,j}$ for every $\alpha, \beta \in \Delta$, where $(,)$ is the killing form on the span of Δ. Let $\delta = \sum_{\alpha \in \Delta} \lambda_\alpha$ and for every $\beta \in \Phi$ let $ht(\beta) := (\beta, \delta)$ be the height of β.

Given a weight λ, we say that λ is *singular* if there exist a positive root α such that $(\alpha, \lambda) = 0$. If λ is not singular, we say that it is *regular of index* p, where p is the number of positive roots α for wich $(\alpha, \lambda) < 0$.

Theorem 2.1. *(Bott* [3]*, theorems* iv *and* iv'*) Let $\mathcal{F}$ be an homogeneous vector bundle over $\mathbf{G}/\mathbf{P}$, which is defined by an irreducible representation of $\mathbf{P}$ with highest weight λ.*
1) If $\lambda + \delta$ is singular, then $H^k(\mathbf{G}/\mathbf{P}, \mathcal{F}) = 0$ for all k,
2) If $\mathcal{F}$ is regular of index p, then $H^k(\mathbf{G}/\mathbf{P}, \mathcal{F}) = 0$ for every $k \neq p$.

$\Diamond$

Remark 2.2. *If the maximal parabolic subgroup $\mathbf{P}$ is generated by $\{\alpha_j\}$, then the ample line bundle $\mathcal{L}$ which generates $Pic(\mathbf{G}/\mathbf{P})$ is defined by an irreducible representation of $\mathbf{P}$ with highest weight λ_j.*

Let Φ_c is the set of positive compact roots as before and let

$$W_1 = \{\omega \in W \mid \omega^{-1}\alpha > 0 \text{ for all } \alpha \in \Phi_c\},$$

where W is the Weyl group of $\mathbf{G}$. Define

$$W_1(q) := \{\omega \in W_1 \mid length(\omega) = q\}.$$

The highest weights of the fully reducible $\mathbf{P}$-module $\Omega^q_{\mathbf{G}/\mathbf{P}}(k)$ occur with multiplicity one and are precisely the weights of the form

$$\omega\delta - \delta + k\lambda_j, \qquad \omega \in W_1(q),$$

so, in order to determine which cohomology groups $H^i(\mathbf{G}/\mathbf{P}, \Omega^q(k))$ vanish, we only need to check whether $\omega\delta + k\lambda_j$ is singular or not, and to compute its index of regularity when it is not.

We can further simplify the problem observing that for every positive compact root α one has $(\alpha, \lambda_j) = 0$, therefore

$$(\omega\delta + k\lambda_i, \alpha) = (\omega\delta, \alpha) = (\delta, \omega^{-1}\alpha) > 0$$

since $\alpha \in \Phi_c$ and $\omega \in W_1$.

Now, if α is a positive non-compact root, then α involves α_j with coefficient 1 and $(\lambda_j, \alpha) = (\lambda_j, \alpha_j) = (\alpha_j, \alpha_j)/2 = c$, thus $\omega\delta + k\lambda_j$ is singular if and only if there is an $\alpha \in \Phi^+ - \Phi_c$ such that $c\,k = -(\delta, \omega^{-1}\alpha)$.

If $\omega\delta + k\lambda_j$ is not singular, then its index of regularity is

$$p = |\ \{\alpha \in \Phi^+ - \Phi_c \mid ck < -(\delta, \omega^{-1}\alpha)\}\ |.$$

Remark 2.3. *$c = 1$, except when $\mathbf{G}$ is of type BI or CI, where $c = 2$.*

Remark 2.4. *With the above notation, since $\omega_{\mathbf{G}/\mathbf{P}}$ is a line bundle, it is an irreducible P-module associated to the fundamental weight $\delta - \sum_{\alpha\in\Phi_c} \alpha = -d_0\lambda_j$.*

It is not hard to prove that

Lemma 2.5. *If $\mu = max\,\{\dfrac{(\alpha,\delta)}{c} \mid \alpha \in \Phi^+\}$ and $k > \mu$ or $k > q$, then*

$$H^i(\mathbf{G}/\mathbf{P}, \Omega^q(k)) = 0$$

for all $i > 0$.

3. Simple coverings

Let $\mathcal{L}$ be an ample divisor on X and consider the varieties $S := \mathrm{Spec}_{\mathcal{O}_X}(\oplus_{i=0}^{\infty}\mathcal{L}^{-i})$ and $\bar{S} = \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}^{-1})$. We say that a smooth variety Z is a simple covering of X of degree N if there is a finite map $f : Z \to X$ of degree N and an embedding $Z \xrightarrow{i} S \hookrightarrow \bar{S}$ for some ample line bundle $\mathcal{L}$ which makes the following diagram commute:

$$\begin{array}{ccccc} Z & \xrightarrow{i} & S & \hookrightarrow & \bar{S} \\ & f \searrow & \pi \downarrow & \swarrow \bar{\pi} & \\ & & X & & \end{array}$$

where π and $\bar{\pi}$ are the natural projections onto X.

Lemma 3.1. *([12], lemma 1.2) If $f : Z \longrightarrow X$ is a simple covering of degree N, then*
1) $f_*\mathcal{O}_Z = \oplus_{i=0}^{N-1}\mathcal{L}^{-i}$,
2) $\omega_{Z/X} = f^*\mathcal{L}^{N-1}$.

Lemma 3.2. *([12], lemma 2.2) Let Z be an smooth projective variety. The following are equivalent:*
1) Z is a simple covering of X of degree N with respect to the ample line bundle $\mathcal{L}$.
2) Z is isomorphic to the zero set of a section in $H^0(\bar{S}, \mathcal{O}_{\bar{S}}(N) \otimes \bar{\pi}^*\mathcal{L}^N)$.

Adjunction formula gives us also $\omega_Z = f^*(\mathcal{L}^{N-1} \otimes \omega_X) = f^*(\mathcal{L}^{N-1} \otimes \mathcal{O}_X(-d_0))$, in view of remark 2.4.

The following is a well know result which we state in our particular situation

Lemma 3.3. *([10], Ex. III, 8.4)*

$$R^i\bar{\pi}_*\mathcal{O}_{\bar{S}}(k) = \begin{cases} sym^k(\mathcal{O}_X \oplus \mathcal{L}^{-1}) & \text{if } \ i=0 \ \text{ and } \ k \geq 0, \\ 0 & \text{if } \ i=0 \ \text{ and } \ k<0, \\ 0 & \text{if } \ i=1 \ \text{ and } \ k>-2, \\ (sym^{-k-2}(\mathcal{O}_X \oplus \mathcal{L}^{-1}))^\vee \otimes \mathcal{L} & \text{if } \ i=1 \ \text{ and } \ k \leq -2. \end{cases}$$

Lemma 3.4. *With the same notation as above, if* $\dim X > 1$*, then one has*

$$H^1(\bar{S}, \tau_{\bar{S}/X}) = 0$$

Proof. Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\bar{S}} \longrightarrow \pi^*(\mathcal{O}_{\bar{S}} \oplus \mathcal{L}) \otimes \mathcal{O}(1) \longrightarrow \tau_{\bar{S}/X} \longrightarrow 0 \tag{3.1}$$

This sequence induces a long exact sequence of cohomology

$$\cdots \longrightarrow H^1(\bar{S}, \pi^*(\mathcal{O} \oplus \mathcal{L}) \otimes \mathcal{O}(1)) \longrightarrow H^1(\bar{S}, \tau_{\bar{S}/X}) \longrightarrow H^2(\bar{S}, \mathcal{O}) \longrightarrow \cdots$$

But $R^1\pi_*\mathcal{O}(k) = 0$ for $k \geq 0$, so using the Leray spectral sequence together with the projection formula and lemma 3.3 one gets:

$$\begin{aligned} H^1(\bar{S}, \pi^*(\mathcal{O} \oplus \mathcal{L}) \otimes \mathcal{O}(1)) &= H^1(X, (\mathcal{O} \oplus \mathcal{L}) \otimes R^0\bar{\pi}_*\mathcal{O}(1)) \\ &= H^1(X, (\mathcal{O} \oplus \mathcal{L}) \otimes (\mathcal{O} \oplus \mathcal{L}^{-1})) \\ &= H^1(X, \mathcal{O} \oplus \mathcal{L} \oplus \mathcal{L}^{-1} \oplus \mathcal{O}) \end{aligned}$$

and

$$H^2(\bar{S}, \mathcal{O}_{\bar{S}}) = H^2(X, \mathcal{O}_X)$$

and the result follows from Bott's theorem, since $\mathcal{L}$ is an ample line bundle Q.E.D.

Lemma 3.5.

$$H^1(\bar{S}, \bar{\pi}^*\tau_X) = 0$$

Proof. Again, Leray spectral sequence together with projection formula and lemma 3.3 give us

$$H^1(\bar{S}, \bar{\pi}^*\tau_X) = H^1(X, \tau_X)$$

and the results follows from Bott's theorem since the fundamental weight associated to τ_X has regularity $\neq 1$ Q.E.D.

Lemma 3.6.

$$H^1(\bar{S}, \tau_{\bar{S}}) = 0$$

Proof. The result follows from 3.4 and 3.5 together with the long exact sequence of cohomology associated to the short exact sequence

$$0 \longrightarrow \tau_{\bar{S}/X} \longrightarrow \tau_{\bar{S}} \longrightarrow \pi^*\tau_X \longrightarrow 0 \tag{3.2}$$

Q.E.D.

Lemma 3.7.

$$H^2(\bar{S}, \tau_{\bar{S}} \otimes \mathcal{O}([-Z])) = 0$$

Proof. By 3.2, $\mathcal{O}([Z]) \cong \mathcal{O}(N) \otimes \bar{\pi}^*\mathcal{L}^N$, therefore

$$H^2(\bar{S}, \tau_{\bar{S}} \otimes \mathcal{O}([-Z])) = H^2(\bar{S}, \tau_{\bar{S}} \otimes \mathcal{O}(-N) \otimes \bar{\pi}^*\mathcal{L}^{-N})$$

Tensoring the short exact sequence (4.3) with $\mathcal{O}(-N) \otimes \bar{\pi}^*\mathcal{L}^{-N}$ one sees that the vanishing of $H^2(\bar{S}, \tau_{\bar{S}} \otimes \mathcal{O}([-Z]))$ follows from that of $H^2(\bar{S}, \tau_{\bar{S}/X} \otimes \mathcal{O}(-N) \otimes \bar{\pi}^*\mathcal{L}^{-N})$ and $H^2(\bar{S}, \bar{\pi}^*(\tau_X) \otimes \mathcal{O}(-N) \otimes \bar{\pi}^*\mathcal{L}^{-N})$.

Projection formula together with 3.3 says that

$$\begin{aligned} H^2(\bar{S}, \bar{\pi}^*(\tau_X) \otimes \mathcal{O}(-N) \otimes \bar{\pi}^*\mathcal{L}^{-N}) &= H^1(X, \tau_X \otimes \mathcal{L}^{-N} \otimes R^1\bar{\pi}_*\mathcal{O}(-N)) \\ &= H^1(X, \tau_X \otimes \mathcal{L}^{-N} \otimes (sym^{N-2}(\mathcal{O}_X \oplus \mathcal{L}^{-1}))^\vee \otimes \mathcal{L}) \end{aligned}$$

Now the vanishing of this group follows from the vanishing of the groups

$$H^1(X, \tau_X \otimes \mathcal{L}^j) \qquad \text{for} \quad -N < j < -1;$$

which in turn are zero because of Bott's theorem.

A similar argument (short exact sequence 3.1 together with projection formula and lemma 3.3) shows that $H^2(\bar{S}, \tau_{\bar{S}/X} \otimes \mathcal{O}(-N) \otimes \bar{\pi}^* \mathcal{L}^{-N}) = 0$. Q.E.D.

4. Kuranishi space of deformations of cyclic coverings

The infinitesimal Torelli theorem for cyclic coverings could be true for trivial reasons, i.e., if the corresponding Kuranishi space is trivial or discrete. In this section we will show that it is not the case in general. In doing so, we closely follows an argument by Wavrik [18], who showed the non triviality of the Kuranishi space of deformations for general cyclic coverings of the proyective space.

Let $X = \mathbf{G}/\mathbf{P}$ with $\mathbf{P}$ a maximal parabolic subgroup, Z be a simple covering of X as before and B be the Kuranishi space of deformations of Z. Then $Z \subset \bar{S} = \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}^{-1})$ is a simple covering of X as in the previous section and we have short exac sequences

$$o \longrightarrow \mathcal{O}_{\bar{S}}(-Z) \longrightarrow \mathcal{O}_{\bar{S}} \longrightarrow i_* \mathcal{O}_Z \longrightarrow 0 \tag{4.1}$$

$$0 \longrightarrow \tau_Z \longrightarrow \tau_{\bar{S}} \otimes \mathcal{O}_Z \longrightarrow \mathcal{N} \longrightarrow 0 \tag{4.2}$$

and

$$o \longrightarrow \tau_{\bar{S}/X} \longrightarrow \tau_{\bar{S}} \longrightarrow \pi^* \tau_X \longrightarrow 0 \tag{4.3}$$

where τ stands for the tangent bundle and $\mathcal{N}$ stands for the normal bundle.

Equation 4.1 gives rice, after tensoring with $\tau_{\bar{S}}$, to the short exac sequence

$$0 \longrightarrow \tau_{\bar{S}} \otimes \mathcal{O}([-Z]) \longrightarrow \tau_{\bar{S}} \longrightarrow i_*(\tau_{\bar{S}} \otimes \mathcal{O}_Z) \longrightarrow 0 \tag{4.4}$$

Associated to 4.2 and 4.4 there are long exact sequences

$$(4.5) \qquad \cdots \longrightarrow H^0(Z,\mathcal{N}) \longrightarrow H^1(Z,\tau_Z) \longrightarrow H^1(Z,\tau_{\bar{S}}\otimes\mathcal{O}_Z) \longrightarrow \cdots$$

$$(4.6) \cdots \to H^1(\bar{S},\tau_{\bar{S}}) \to H^1(\bar{S},i_*(\tau_{\bar{S}}\otimes\mathcal{O}_Z)) \to H^2(\bar{S},\tau_{\bar{S}}\otimes\mathcal{O}([-Z])) \to \cdots$$

We would like to have $H^1(Z,\tau_{\bar{S}}\otimes\mathcal{O}_Z) = 0$ in 4.5, in order to compute the dimension of $H^1(Z,\tau_Z)$. Since $H^1(Z,\tau_{\bar{S}}\otimes\mathcal{O}_Z) = H^1(\bar{S},i_*(\tau_{\bar{S}}\otimes\mathcal{O}_Z))$, by 4.6, we need to show that

$$H^1(\bar{S},\tau_{\bar{S}}) = H^2(\bar{S},\tau_{\bar{S}}\otimes\mathcal{O}([-Z])) = 0$$

which is the content of 3.6 and 3.7, so $H^1(Z,\tau_{\bar{S}}\otimes\mathcal{O}_Z) = 0$.

Therefore we have a short exact sequence

$$H^0(Z,\tau_{\bar{S}}\otimes\mathcal{O}_Z) \longrightarrow H^0(Z,\mathcal{N}) \longrightarrow H^1(Z,\tau_Z) \longrightarrow 0$$

and $\dim H^1(Z,\tau_Z) > 0$ provided the first map is not surjective. The following propositions tell us that this will be the case in general and are due to Wavrik (see [18]), though he stated them only for $\mathbf{G}/\mathbf{P} = \mathbb{P}^n$, but the proofs are the same, *mutatis mutandis*.

Proposition 4.1.

$$h^0(Z,\mathcal{N}) = \sum_{j=0}^{N} h^0(\mathbf{G}/\mathbf{P},\mathcal{L}^j) - 1.$$

Proposition 4.2. $H^0(Z,\tau_{\bar{S}}\otimes\mathcal{O}_Z) = 0$ *if* $d(N-1) > d_0$*, where* $d = c_1(\mathcal{L})$ *and* d_0 *is the constant appearing in remark 2.4, which for* $\mathbf{G}$ *of type* A_n *is equal to* $n+1$.

In conclusion, in the general situation the Kuranishi space of deformations of cyclic coverings of $\mathbf{G}/\mathbf{P}$ is not discrete.

5. Infinitesimal Torelli for cyclic coverings of $\mathbf{G}/\mathbf{P}$.

Let $X = \mathbf{G}/\mathbf{P}$ with $\mathbf{P}$ a maximal parabolic subgroup, Z be a simple covering of X as before and B be the Kuranishi space of deformations of Z. Suppose B is smooth and let $m : B \to D$ be the local period map into the period domain D of all Hodge structures on $H^n(Z,\mathbb{C})$, where $n = \dim Z$. The map m is holomorphic, with tangent map at the point Z:

$$T : H^1(Z,\tau_Z) \to T_{m(0)}D = \oplus_p \mathrm{Hom}(H^{n-p}(Z,\Omega_Z^p), H^{n-p+1}(Z,\Omega_Z^{p-1}))$$

We say that the infinitesimal Torelli theorem holds for Z if the tangent map T is injective.
Obviously it will be enought to show that there is a p such that the map

$$T_p : H^1(Z, \tau_Z) \to \mathrm{Hom}(H^p(Z, \Omega_Z^{n-p}), H^{p+1}(Z, \Omega_Z^{n-p-1}))$$

obtained from T and the natural projection into the p-factor, is injective. In particular, it will suffy to show it for $p = 0$ or, what amounts to the same because of Serre's duality, it will be enough to show that the map

$$H^0(Z, \omega_Z) \otimes H^{n-1}(Z, \Omega_Z^1) \longrightarrow H^{n-1}(Z, \Omega_Z^1 \otimes \omega_Z) \tag{5.1}$$

is surjective, where as usual, $\omega_Y = \Omega_Y^n$ for any smooth n-dimensional variety Y.

For Z smooth, the exact sequence

$$0 \longrightarrow \mathcal{O}_Z(-Z) \longrightarrow \Omega^1_{\bar{S}|Z} \xrightarrow{\beta} \Omega_Z^1 \longrightarrow 0 \tag{5.2}$$

induces exact sequences

$$0 \longrightarrow \Omega_Z^{n-p} \otimes \mathcal{O}_Z(-Z) \xrightarrow{\alpha} \Omega^{n-p+1}_{\bar{S}|Z} \xrightarrow{\beta} \Omega_Z^{n-p+1} \longrightarrow 0\ , \tag{5.3}$$

which for every locally free $\mathcal{O}_Z$-module $\mathcal{G}$ give rise, after tensoring with an appropriate multiple of $\mathcal{O}_Z(Z)$, to a complex $(K^\bullet, d_\bullet)$, where

$$K^j := \Omega^{n-p+j+1}_{\bar{S}|Z} \otimes \mathcal{O}_Z((j+1)Z) \otimes \mathcal{G} \tag{5.4}$$

and the derivatives are induced by $\alpha \circ \beta$. This complex is exact, except at $j = 0$, where $\ker\ d_0 = \Omega_Z^{n-p} \otimes \mathcal{G}$, therefore the complex $(K^\bullet, d_\bullet)$ is quasi-isomorphic to $\Omega_Z^{n-p} \otimes \mathcal{G}$.
Moreover, the exact sequence 5.2 implies $K^p = \omega_Z \otimes \mathcal{O}_Z(pZ) \otimes \mathcal{G}$ and $K^j = 0$ for $j > p$.

Observe that $K^p[-p] = F^p K^\bullet$, so that we have a map $\mathbb{H}^p(Z, K^p[-p]) \xrightarrow{\gamma} \mathbb{H}^p(Z, K^\bullet)$.

Following [12], let us define the map $\gamma_p(\mathcal{G})$ as the composition

$$H^0(Z, \omega_Z \otimes \mathcal{O}_Z(pZ) \otimes \mathcal{G}) \cong \mathbb{H}^p(K^p[-p]) \xrightarrow{\gamma} \mathbb{H}^p(K^\bullet) \cong H^p(Z, \Omega_Z^{n-p} \otimes \mathcal{G}).$$

The following lemma gives us sufficient conditions for the map 5.1 to be surjective.

Lemma 5.1 ([12])**.** *If the map*

$$H^0(Z, \omega_Z^2 \otimes \mathcal{O}_Z((n-1)Z)) \xrightarrow{\gamma_{n-1}(\omega_Z)} H^{n-1}(Z, \Omega_Z^1 \otimes \omega_Z)$$

and the map

$$H^0(Z, \omega_Z) \otimes H^0(Z, \omega_Z \otimes \mathcal{O}_Z((n-1)Z)) \longrightarrow H^0(Z, \omega_Z) \otimes H^{n-1}(Z, \Omega_Z^1)$$

are surjective, then the map 5.1 is surjective and the infinitesimal Torelli theorem holds for Z.

Proof. One has a commutative diagram

$$\begin{array}{ccc} H^0(Z,\omega_Z)\otimes H^{n-1}(Z,\Omega^1_Z) & \longrightarrow & H^{n-1}(Z,\Omega^1_Z\otimes\omega_z) \\ id\otimes\gamma_{n-1}(\mathcal{O}_Z)\uparrow & & \uparrow id\otimes\gamma_{n-1}(\omega_Z) \\ H^0(Z,\omega_Z)\otimes H^0(Z,\omega_z\otimes\mathcal{O}_Z((n-1)Z) & \xrightarrow{\eta} & H^0(Z,\omega_Z^2\otimes\mathcal{O}_Z((n-1)Z)) \end{array} \tag{5.5}$$

and the assumptions of the lemma imply the surjectivity of the map in the top row.

Theorem 5.2. *Let $X=\mathbf{G/P}$, $Z\longrightarrow X$ be a simple covering of degree N with respect to $\mathcal{O}_X(d)$ and d_0,μ be as in remark 2.4 and lemma 2.5. If $d(N-1)-d_0>\mu$ or $d(N-1)-d_0>n-1$, then the infinitesimal Torelli theorem holds for Z.*

Proof. We will freely use the notation of section 3, with $\mathcal{L}=\mathcal{O}_X(d)$. Remember that $n=dim\ X$.

According with 5.1, it is enough to show the surjectivity of the maps

$$H^0(Z,\omega_Z^2\otimes\mathcal{O}_Z((n-1)Z))\overset{\gamma_{n-1}(\omega_Z)}{\longrightarrow}H^{n-1}(Z,\Omega^1_Z\otimes\omega_Z)$$

and

$$H^0(Z,\omega_Z)\otimes H^0(Z,\omega_Z\otimes\mathcal{O}_Z((n-1)Z))\longrightarrow H^0(Z,\omega_Z)\otimes H^{n-1}(Z,\Omega^1_Z)$$

I) The surjectivity of $\gamma_p(\omega_Z)$ follows from the spectral sequence $H^i(Z,K^j)\Longrightarrow\mathcal{H}^{i+j}(K^\bullet)$ as soon as $H^{s+1}(Z,K^{p-s-1})=0$ for $s=0,\cdots,p-1$. In our case $p=n-1$, $K^j:=\Omega^{n-p+j+1}_{\bar{S}|Z}\otimes\mathcal{O}_Z((j+1)Z)\otimes\mathcal{G}$ and $\mathcal{G}=\omega_Z$, in other words, we need to show that

$$H^{s+1}(Z,\Omega^{n-s}_{\bar{S}|z}\otimes\mathcal{O}_Z((n-1-s)Z)\otimes\omega_Z)=0$$

for $s=0,\cdots,n-2$.
The classical short exact sequence for the relative differentials

$$0\longrightarrow\bar{\pi}^*\Omega^1_X\longrightarrow\Omega^1_{\bar{S}}\longrightarrow\Omega^1_{\bar{S}/X}\longrightarrow 0$$

induces, after exterior product and restriction to Z, the short exact sequence

$$0\longrightarrow f^*\Omega^{n-s}_X\longrightarrow\Omega^{n-s}_{\bar{S}|Z}\longrightarrow f^*(\Omega^{n-s-1}_X\otimes\mathcal{L}^{-1})\longrightarrow 0$$

for every s, which in turn produces, after tensoring with $\mathcal{O}_Z((n-1-s)Z)\otimes\omega_Z$, the exact sequence

$$\begin{array}{l} 0\longrightarrow f^*(\Omega^{n-s}_X\otimes\mathcal{L}^{(n-s)\cdot N-1}\otimes\omega_X)\longrightarrow\Omega^{n-s}_{\bar{S}|Z}\otimes\mathcal{O}_Z((n-1-s)Z)\otimes\omega_Z \\ \qquad\qquad\longrightarrow f^*(\Omega^{n-s-1}_X\otimes\mathcal{L}^{(n-s)\cdot N-2}\otimes\omega_X)\longrightarrow 0. \end{array}$$

Projection formula together with lemma 3.1 gives

$$H^{s+1}(Z, f^*(\Omega_X^{n-s} \otimes \mathcal{L}^{(n-s)\cdot N-1} \otimes \omega_X)) = \oplus_{i=0}^{N-1} H^{s+1}(X, \Omega_X^{n-s} \otimes \mathcal{L}^{(n-s)\cdot N-1-i} \otimes \omega_X)$$

and also

$$H^{s+1}(Z, f^*(\Omega_X^{n-s-1} \otimes \mathcal{L}^{(n-s)\cdot N-2} \otimes \omega_X)) = \oplus_{i=0}^{N-1} H^{s+1}(X, \Omega_X^{n-s-1} \otimes \mathcal{L}^{(n-s)\cdot N-2-i} \otimes \omega_X)$$

Observe that $n-s \geq 2$ and $0 \leq i \leq N-1$, so that $(n-s)N-1-i \geq (n-s)N-2-i \geq N-1$. Since $\mathcal{L}^j = \mathcal{O}_X(jd)$ for every j, then $d[(n-s)N-2-i] \geq d(N-1)$. Finally, as observed in remark 2.4, $\omega_X = \mathcal{O}_X(-d_0)$, therefore the assumption on d and N, together with proposition 2.5 give us the vanishing of this cohomology groups and therefore the vanishing of

$$H^{s+1}(Z, \Omega_{\bar{S}|z}^{n-s} \otimes \mathcal{O}_Z((n-1-s)Z) \otimes \omega_Z)$$

which implies the surjectivity of $\gamma_{n-1}(\omega_Z)$, as desired.

II) Let us now consider the map

$$H^0(Z, \omega_Z) \otimes H^0(Z, \omega_Z \otimes \mathcal{O}_Z((n-1)Z)) \longrightarrow H^0(Z, \omega_Z) \otimes H^{n-1}(Z, \Omega_Z^1)$$

The surjectivity of this map will follows from the surjectivity of the map

$$H^0(Z, \omega_Z \otimes \mathcal{O}_Z((n-1)Z)) \longrightarrow H^{n-1}(Z, \Omega_Z^1).$$

Now, for every p we have short exact sequences

$$0 \longrightarrow \Omega_Z^{n-p-1}(-(k+1)Z) \longrightarrow \Omega_{S|Z}^{n-p}(-kZ) \longrightarrow \Omega_Z^{n-p}(-kZ) \longrightarrow 0,$$

which induce long exact sequences

$$\cdots \longrightarrow H^p\left(\Omega_Z^{n-p}(-(n-p-1)Z)\right) \xrightarrow{\beta_p} H^{p+1}\left(\Omega_Z^{n-p-1}(-(n-p-2)Z)\right) \longrightarrow$$
$$\longrightarrow H^{p+1}\left(\Omega_{S|Z}^{n-p}(-(n-p-1)Z)\right) \longrightarrow \cdots ,$$

so the surjectivity of β_p for every p will give us the surjectivity we are looking for, but this will be the case if $H^{p+1}\left(\Omega_{S|Z}^{n-p}(-(n-p-1)Z)\right) = 0$ for every p, which follows from the computations on I), since $d_0 \geq 1$ always.

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